

CLASSIFYING $\mathrm{GL}(n, \mathbb{Z})$ -ORBITS OF POINTS AND RATIONAL SUBSPACES

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ABSTRACT. We first show that the subgroup of the abelian real group \mathbb{R} generated by the coordinates of a point in $x \in \mathbb{R}^n$ completely classifies the $\mathrm{GL}(n, \mathbb{Z})$ -orbit of x . This yields a short proof of J.S.Dani's theorem: the $\mathrm{GL}(n, \mathbb{Z})$ -orbit of $x \in \mathbb{R}^n$ is dense iff $x_i/x_j \in \mathbb{R} \setminus \mathbb{Q}$ for some $i, j = 1, \dots, n$. We then classify $\mathrm{GL}(n, \mathbb{Z})$ -orbits of rational affine subspaces F of \mathbb{R}^n . We prove that the dimension of F together with the volume of a special parallelotope associated to F yields a complete classifier of the $\mathrm{GL}(n, \mathbb{Z})$ -orbit of F .

1. INTRODUCTION

Throughout we let $\mathrm{GL}(n, \mathbb{Z})$ denote the group of linear transformations of the form $x \mapsto \mathcal{U}x$ for $x \in \mathbb{R}^n$, where \mathcal{U} is an integer $(n \times n)$ -matrix with $\det(\mathcal{U}) = \pm 1$. We let $\mathrm{orb}(x) = \{\gamma(x) \in \mathbb{R}^n \mid \gamma \in \mathrm{GL}(n, \mathbb{Z})\}$ denote the $\mathrm{GL}(n, \mathbb{Z})$ -orbit of $x \in \mathbb{R}^n$.

For all $n = 1, 2, \dots$ we classify $\mathrm{GL}(n, \mathbb{Z})$ -orbits of points in \mathbb{R}^n : every point in \mathbb{R}^n is assigned an invariant, in such a way that two points have the same $\mathrm{GL}(n, \mathbb{Z})$ -orbit iff they have the same invariant. For each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, the invariant is given by the group

$$H_x = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \dots + \mathbb{Z}x_n$$

generated by x_1, x_2, \dots, x_n in the additive group \mathbb{R} . As a first application we give a short self-contained proof of the characterization [2, Theorem 17] of those $x \in \mathbb{R}^n$ having a dense $\mathrm{GL}(n, \mathbb{Z})$ - and $\mathrm{SL}(n, \mathbb{Z})$ -orbit. Our proof is also shorter and more elementary than the one given in [3, Corollary 3.1].

A *rational affine hyperplane* $H \subseteq \mathbb{R}^n$ is a set of the form $H = \{z \in \mathbb{R}^n \mid \langle h, z \rangle = r\}$, for some nonzero vector $h \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$. Here $\langle -, - \rangle$ denotes scalar product. When $r = 0$ we say that H is a *rational linear hyperplane*. A *rational affine (resp., linear) subspace* A of \mathbb{R}^n is an intersection of rational affine (resp., linear) hyperplanes in \mathbb{R}^n .

In Theorem 11 we classify $\mathrm{GL}(n, \mathbb{Z})$ -orbits of rational affine subspaces of \mathbb{R}^n .

With respect to the vast literature on orbits of discrete groups acting on \mathbb{R}^n [2, 5, 7, 11, 12, 15], our results highlight the crucial role of (Farey) regularity of simplicial complexes, i.e., the regularity (=nonsingularity=smoothness) of their associated fans and toric varieties [6, 13]. Regular simplicial complexes were also used in [1] to classify $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ -orbits of points.

2. CLASSIFICATION OF $\mathrm{GL}(n, \mathbb{Z})$ -ORBITS OF POINTS AND A SHORT PROOF OF J.S.DANI'S THEOREM

We first show that the group H_x classifies the $\mathrm{GL}(n, \mathbb{Z})$ -orbit of $x \in \mathbb{R}^n$.

Proposition 1. *For all $x, y \in \mathbb{R}^n$, $H_y = H_x$ iff $y = \gamma(x)$ for some $\gamma \in \mathrm{GL}(n, \mathbb{Z})$.*

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Proof. The (\Leftarrow) -direction is trivial. For the converse (\Rightarrow) -direction, let $L_x = \bigcap \{L \subseteq \mathbb{R}^n \mid x \in L \text{ and } L \text{ is a rational linear subspace of } \mathbb{R}^n\}$. Let $e = \dim(L_x)$. Let v_1, \dots, v_n be a basis of the free abelian group $\mathbb{Z}^n \subseteq \mathbb{R}^n$, with $v_1, \dots, v_e \in L_x$. Let the map $\alpha \in \text{GL}(n, \mathbb{Z})$ send each v_i to the standard i th basis vector ϵ_i of the vector space \mathbb{R}^n . Let $x' = \alpha(x)$. Then $x' \in \alpha(L_x) = \{r \in \mathbb{R}^n \mid r_{e+1} = \dots = r_n = 0\}$. It follows that $e \geq \text{rank}(H_x)$. The assumed minimality of L_x yields $e \leq \text{rank}(H_x)$, whence $e = \text{rank}(H_x)$. Trivially, $H_{x'} = H_x$, and $\text{rank}(H_{x'}) = e$. The first e coordinates x'_1, \dots, x'_e of x' are a basis of the group $H_{x'}$.

Now, suppose $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ satisfies $H_y = H_x$, with the intent of showing that $y \in \text{orb}(x)$. Then $e = \dim(L_x) = \text{rank}(H_x) = \text{rank}(H_y) = \dim(L_y)$. As above, let w_1, \dots, w_n be a basis of \mathbb{Z}^n , with $w_1, \dots, w_e \in L_y$. Let $\beta \in \text{GL}(n, \mathbb{Z})$ send each w_i to ϵ_i . Let $y' = \beta(y)$. The first e coordinates y'_1, \dots, y'_e of y' are a basis of the group $H_{y'}$. There is $\delta \in \text{GL}(e, \mathbb{Z})$ mapping x'_j to y'_j for each $j = 1, \dots, e$. By appending a diagonal of $n - e$ ones to the matrix of δ , we obtain the matrix of a map $\hat{\delta} \in \text{GL}(n, \mathbb{Z})$ sending $x' = (x'_1, \dots, x'_e, 0, \dots, 0)$ to $y' = (y'_1, \dots, y'_e, 0, \dots, 0)$. The composite map $\gamma = \beta^{-1}\hat{\delta}\alpha \in \text{GL}(n, \mathbb{Z})$ satisfies $\gamma(x) = y$. \square

Proposition 2. *For any $x \in \mathbb{R}^n$ let $e = \text{rank}(H_x)$ and \mathcal{B}_x be the set of all (ordered) bases $b = (b_1, \dots, b_e)$ of H_x . Suppose $\emptyset = \mathbb{R}x \cap \mathbb{Z}^n$, where $\mathbb{R}x = \{\rho x \mid \rho \in \mathbb{R}\}$. Then:*

- (a) *The origin $0 \in \mathbb{R}^e$ is an accumulation point of the set \mathcal{B}_x , identified with a subset of \mathbb{R}^e .*
- (b) *More generally, for each $i = 1, \dots, e$, every point on the i th axis $\mathbb{R}\epsilon_i$ of \mathbb{R}^e is an accumulation point of \mathcal{B}_x .*
- (c) *\mathcal{B}_x is dense in \mathbb{R}^e .*
- (d) *Both $\text{orb}(x)$ and the $\text{SL}(n, \mathbb{Z})$ -orbit of x are dense in \mathbb{R}^n .*

Proof. By assumption $n \geq \text{rank}(H_x) \geq 2$. Further, for no integer vector $p \in \mathbb{Z}^n$ the point x belongs to the set $\mathbb{R}_{\geq 0}p = \{\rho p \mid 0 \leq \rho \in \mathbb{R}\}$.

(a) Pick $g = (g_1, \dots, g_e) \in \mathcal{B}_x$. Fix $i \in \{1, \dots, e\}$. Our assumption about x yields an index $j = j_i$ with $g_i/g_j \notin \mathbb{Q}$. Then the euclidean algorithm yields a sequence g_{it}, g_{jt} of real numbers such that $\lim_{t \rightarrow \infty} g_{it} = 0 = \lim_{t \rightarrow \infty} g_{jt}$, and for each t the elements g_{it}, g_{jt} together with the remaining g_k ($k \neq i, j$) of g form a basis of H_x . Proceeding inductively on i we obtain a sequence of ordered bases g_t of H_x such that $\lim_{t \rightarrow \infty} g_t = (0, 0, \dots, 0) \in \mathbb{R}^e$.

(b) It suffices to argue for $i = 1$. There is $j = 2, \dots, e$ such that $g_1/g_j \notin \mathbb{Q}$, say $j = 2$ without loss of generality. Part (a) yields a sequence of bases $g_t = (g_{1t}, \dots, g_{et})$ converging to the origin of \mathbb{R}^e . Let z be an arbitrary point on the first axis $\mathbb{R}\epsilon_1$ of \mathbb{R}^e . For some $\xi \in \mathbb{R}$ we may write $z = \xi\epsilon_1$. Let $B_\epsilon(z) \subseteq \mathbb{R}^e$ be the open ball of radius $\epsilon > 0$ centered at z . Then for all suitably large t there is an integer k_t such that the point $(k_t g_{1t} + g_{1t}, g_{2t}, \dots, g_{et})$ belongs to $B_\epsilon(z)$.

(c) Fix $u \in \mathbb{R}^e$ and $\epsilon > 0$. Again let $B_\epsilon(u) \subseteq \mathbb{R}^e$ be the open ball of radius $\epsilon > 0$ centered at u . In view of (a) it suffices to argue in case $u \neq 0$. Then there is a vector $q \in \mathbb{Z}^e$ and a point z lying in $\mathbb{R}_{>0}q \cap B_\epsilon(u)$. (The intersection of the sphere $S^{e-1} \subseteq \mathbb{R}^e$ with the set of rational halflines in \mathbb{R}^e with vertex at the origin, is a dense subset of S^{e-1}). We may insist that q is *primitive* i.e., the gcd of its coordinates is 1. Since q can be extended to a basis of the free abelian group \mathbb{Z}^n then some $\delta \in \text{GL}(e, \mathbb{Z})$ maps q to the first basis vector ϵ_1 of the vector space \mathbb{R}^e . For some $0 < \omega \in \mathbb{R}$ we can write $\delta(z) = \omega\epsilon_1$. As proved in (b), $\omega\epsilon_1$ is an accumulation point of the set \mathcal{B}_x of bases $b = (b_1, \dots, b_e)$ of H_x . Since δ^{-1} , as well as δ , is continuous and one-one, z is an accumulation point of the set Δ of δ^{-1} -images of these bases. Thus some basis (h_1, \dots, h_e) of H_x belongs to $B_\epsilon(u)$.

(d) Let $h = (h_1, \dots, h_e, 0, \dots, 0) \in \mathbb{R}^n$. Then $H_h = H_x$. By Proposition 1, $h \in \text{orb}(x)$. Since $e \geq 2$ (by possibly exchanging h_1 and h_2) we may insist that

there exists $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ such that $\gamma(x) = h$. Identifying \mathbb{R}^e with the subspace of \mathbb{R}^n of points whose last $n - e$ coordinates are zero, we see that $\mathrm{orb}(x)$ is dense in \mathbb{R}^e . Let L be an arbitrary rational e -dimensional linear subspace L of \mathbb{R}^n . Since $e \geq 2$, L is the η -image of \mathbb{R}^e for some $\eta \in \mathrm{SL}(n, \mathbb{Z}) \subseteq \mathrm{GL}(n, \mathbb{Z})$. Thus $\mathrm{orb}(x)$ is dense in L as well. We conclude that $\mathrm{orb}(x)$ and the $\mathrm{SL}(n, \mathbb{Z})$ -orbit of x are dense in \mathbb{R}^n . \square

Corollary 3. [2, Theorem 17] *The $\mathrm{GL}(n, \mathbb{Z})$ -orbit (equivalently, the $\mathrm{SL}(n, \mathbb{Z})$ -orbit) of $x \in \mathbb{R}^n$ is dense in \mathbb{R}^n iff $\mathbb{R}x \cap \mathbb{Z}^n = \emptyset$ iff for no $p \in \mathbb{Z}^n$, $x \in \mathbb{R}_{\geq 0}p$.*

Proof. If $x = 0$ we have nothing to prove. Suppose $x \neq 0$ belongs to some rational halfline $\mathbb{R}_{\geq 0}p$, say $x = \xi p$ for some $0 \leq \xi$ and $p \in \mathbb{Z}^n$. We may assume p primitive. Then every $\gamma \in \mathrm{GL}(n, \mathbb{Z})$ will send p to some primitive integer vector $\gamma(p)$ of \mathbb{R}^n , and $\gamma(x)$ will have the form $\xi\gamma(p)$. Since the length of the shortest primitive integer vector in \mathbb{R}^n is 1, it follows that $\mathrm{orb}(x)$ is disjoint from the open ball $B_\xi(0)$. The same argument holds a fortiori for $\mathrm{SL}(n, \mathbb{Z})$ -orbits. The converse follows from Proposition 2(d). \square

3. REGULAR SIMPLEXES AND THE λ_i -MEASURE OF A RATIONAL POLYHEDRON

In this section we present the necessary material for our classification of $\mathrm{GL}(n, \mathbb{Z})$ -orbits of rational affine spaces. We refer the reader to [6, 14] for elementary background on simplicial complexes and polyhedral topology.

For any set $E \subseteq \mathbb{R}^n$ the *affine hull* $\mathrm{aff}(E)$ is the set of all *affine combinations* in \mathbb{R}^n of elements of E .

A finite set $\{z_1, \dots, z_m\}$ of points in \mathbb{R}^n is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements. One then easily sees that a subset R of \mathbb{R}^n is an m -dimension rational affine subspace of \mathbb{R}^n iff there exist affinely independent $v_0, \dots, v_m \in \mathbb{Q}^n$ such that $R = \mathrm{aff}(v_0, \dots, v_m)$. For $0 \leq m \leq n$, an m -simplex in \mathbb{R}^n is the *convex hull* $T = \mathrm{conv}(v_0, \dots, v_m)$ of $m + 1$ affinely independent points $v_0, \dots, v_m \in \mathbb{R}^n$. The *vertices* v_0, \dots, v_m are uniquely determined by T . T is said to be a *rational simplex* if its vertices are rational. A *rational polyhedron* P in \mathbb{R}^n is the union of finitely many rational simplexes T_i in \mathbb{R}^n . P need not be convex or connected. The T_i need not have the same dimension.

By the *denominator* $\mathrm{den}(x)$ of a rational point $x = (x_1, \dots, x_n)$ we understand the least common denominator of its coordinates. The *homogeneous correspondent* of a rational point $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$ is the integer vector

$$\tilde{x} = (\mathrm{den}(x) \cdot x_1, \dots, \mathrm{den}(x) \cdot x_n, \mathrm{den}(x)) \in \mathbb{Z}^{n+1}.$$

Let $n = 1, 2, \dots$ and $m = 0, \dots, n$. Following [10], a rational m -simplex $T = \mathrm{conv}(v_0, \dots, v_m) \subseteq \mathbb{R}^n$ is said to be (*Farey*) *regular* if the set $\{\tilde{v}_0, \dots, \tilde{v}_m\}$ of homogeneous correspondents of the vertices of T can be extended to a basis of the free abelian group \mathbb{Z}^{n+1} .

By a *simplicial complex* in \mathbb{R}^n we mean a finite set Δ of simplexes S_i in \mathbb{R}^n , closed under taking faces, and having the further property that any two elements of Δ intersect in a common face. The complex Δ is said to be *rational* if the vertices of all $S_i \in \Delta$ are rational. For every complex Δ , its *support* $|\Delta| \subseteq \mathbb{R}^n$ is the pointset union of all simplexes of Δ . In this case we say that Δ is a *triangulation* of $|\Delta|$. We say that Δ is *regular* if every simplex of Δ is regular.

3.1. The rational measure λ_d . For fixed $n = 1, 2, \dots$ let $P \subseteq \mathbb{R}^n$ be a (not necessarily rational) polyhedron. For any triangulation Δ of P and $i = 0, 1, \dots$ let $\Delta^{\max}(i)$ denote the set of maximal i -simplexes of Δ . The *i -dimensional part* $P^{(i)}$

of P is now defined by

$$P^{(i)} = \bigcup \{T \mid T \in \Delta^{\max}(i)\}. \quad (1)$$

Since any two triangulations of P have a joint subdivision, the definition of $P^{(i)}$ does not depend on the chosen triangulation Δ of P . If $P^{(i)}$ is nonempty, then it is an i -dimensional polyhedron (i.e., a finite union of i -simplexes in \mathbb{R}^n) whose j -dimensional part $P^{(j)}$ is empty for each $j \neq i$. Trivially, $P^{(k)} = \emptyset$ for each integer $k > \dim(P)$.

For every regular m -simplex $S = \text{conv}(x_0, \dots, x_m) \subseteq \mathbb{R}^n$ we use the notation

$$\text{den}(S) = \prod_{j=0}^m \text{den}(x_j).$$

Let R be a *rational* polyhedron \mathbb{R}^n . In [10, Lemma 2.1] it is proved that R has a regular triangulation. For every regular triangulation Δ of R , and $i = 0, 1, \dots$, the rational number $\lambda(n, i, R, \Delta)$ is defined by

$$\lambda(n, i, R, \Delta) = \begin{cases} \sum_{T \in \Delta^{\max}(i)} \frac{1}{i! \text{den}(T)} & \text{if } \Delta^{\max}(i) \neq \emptyset \\ 0 & \text{if } \Delta^{\max}(i) = \emptyset. \end{cases}$$

As proved in [10, Theorem 2.3], $\lambda(n, i, R, \Delta)$ does not depend on the regular triangulation Δ of R . The proof of this result relies upon the solution of the weak Oda conjecture by Morelli and Włodarczyk [8, 16]. Thus, for any rational polyhedron $R \subseteq \mathbb{R}^n$ and $i = 0, 1, 2, \dots$, we can unambiguously write

$$\lambda_i(R) = \lambda(n, i, R, \Delta),$$

where Δ is an arbitrary regular triangulation of R . We say that $\lambda_i(R)$ is the *i -dimensional rational measure* of R . Trivially, $\lambda_i(R) = 0$ for each integer $i > \dim(R)$.

For a characterization of λ_i let

$$\mathcal{G}_n = \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$$

denote the group of transformations of the form $x \mapsto \mathcal{U}x + b$ ($x \in \mathbb{R}^n$), where $b \in \mathbb{Z}^n$ and \mathcal{U} is an integer $(n \times n)$ -matrix with determinant ± 1 . \mathcal{G}_n is known as the n -dimensional affine group over the integers. Also let $\mathcal{P}^{(n)}$ denote the set of all rational polyhedra in \mathbb{R}^n .

Building on the main result of [9], in [10, 1.1, 4.1, 6.2] the following result is proved:

Theorem 4. *For each $n = 1, 2, \dots$ and $i = 0, 1, \dots$, the map $\lambda_i: \mathcal{P}^{(n)} \rightarrow \mathbb{R}_{\geq 0}$ has the following properties, for all $P, Q \in \mathcal{P}^{(n)}$:*

- (i) **Invariance:** *If $P = \gamma(Q)$ for some $\gamma \in \mathcal{G}_n$ then $\lambda_i(P) = \lambda_i(Q)$.*
- (ii) **Valuation:** *$\lambda_i(\emptyset) = 0$, $\lambda_i(P) = \lambda_i(P^{(i)})$, and the restriction of λ_i to the set of all rational polyhedra P, Q in \mathbb{R}^n having dimension $\leq i$ is a valuation: in other words,*

$$\lambda_i(P) + \lambda_i(Q) = \lambda_i(P \cup Q) + \lambda_i(P \cap Q).$$

- (iii) **Conservativity:** *Let $(P, 0) = \{(x, 0) \in \mathbb{R}^{n+1} \mid x \in P\}$. Then $\lambda_i(P) = \lambda_i(P, 0)$.*

- (iv) **Pyramid:** *For $k = 1, \dots, n$, if $\text{conv}(v_0, \dots, v_k)$ is a regular k -simplex in \mathbb{R}^n with $v_0 \in \mathbb{Z}^n$ then*

$$\lambda_k(\text{conv}(v_0, \dots, v_k)) = \lambda_{k-1}(\text{conv}(v_1, \dots, v_k))/k.$$

(v) **Normalization:** Let $j = 1, \dots, n$. Suppose the set $B = \{w_1, \dots, w_j\} \subseteq \mathbb{Z}^n$ is part of a basis of the free abelian group \mathbb{Z}^n . Let the closed parallelotope $\mathcal{P}_B \subseteq \mathbb{R}^n$ be defined by

$$\mathcal{P}_B = \left\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^j \gamma_i w_i, \ 0 \leq \gamma_i \leq 1 \right\}.$$

Then $\lambda_j(\mathcal{P}_B) = 1$.

(vi) **Proportionality:** Let A be an m -dimensional rational affine subspace of \mathbb{R}^n for some $m = 0, \dots, n$. Then there is a constant $\kappa_A > 0$, only depending on A , such that $\lambda_m(Q) = \kappa_A \cdot \mathcal{H}^m(Q)$ for every rational m -simplex $Q \subseteq A$. Here as usual, \mathcal{H}^m denotes the m -dimensional Hausdorff measure, [4]. Moreover, if $m = n$, then $\kappa_A = 1$.

Conversely, conditions (i)-(vi) uniquely characterize the maps λ_i among all maps $\mu: \mathcal{P}^{(n)} \rightarrow \mathbb{R}_{\geq 0}$.

4. ORBITS OF RATIONAL AFFINE SUBSPACES OF \mathbb{R}^n : PRELIMINARIES

Lemma 5. For each rational point $x \in \mathbb{Q}^n$ and $\psi \in \text{GL}(n, \mathbb{Z})$, $\text{den}(x) = \text{den}(\psi(x))$. Moreover, if $\text{conv}(0, v_1, \dots, v_n) \subseteq \mathbb{R}^n$ and $\text{conv}(0, w_1, \dots, w_n) \subseteq \mathbb{R}^n$ are regular n -simplexes, the following conditions are equivalent:

- (i) $\text{den}(v_i) = \text{den}(w_i)$ for all $i \in \{1, \dots, n\}$;
- (ii) there exists $\gamma \in \text{GL}(n, \mathbb{Z})$ such that $\gamma(v_i) = w_i$ for each $i \in \{1, \dots, n\}$.

Proof. (ii) \Rightarrow (i) is trivial. For the converse direction (i) \Rightarrow (ii), from the regularity of $\text{conv}(0, v_1, \dots, v_n)$ and $\text{conv}(0, w_1, \dots, w_n)$ we obtain bases $\{\tilde{0}, \tilde{v}_1, \dots, \tilde{v}_n\}$ and $\{\tilde{0}, \tilde{w}_1, \dots, \tilde{w}_n\}$ of the free abelian group \mathbb{Z}^n . (Note $\tilde{0} = (0, \dots, 0, 1)$). There are integer $((n+1) \times (n+1))$ -matrices \mathcal{C} and \mathcal{D} such that $\mathcal{C}\tilde{v}_i = \tilde{w}_i$ and $\mathcal{D}\tilde{w}_i = \tilde{v}_i$ for each $i \in \{0, 1, \dots, n\}$. It follows that $\mathcal{C} \cdot \mathcal{D}$ and $\mathcal{D} \cdot \mathcal{C}$ are equal to the identity matrix. For each $i \in \{0, 1, \dots, n\}$ the last coordinate of \tilde{v}_i , as well as the last coordinate of \tilde{w}_i , coincide with $\text{den}(v_i) = \text{den}(w_i)$. Thus \mathcal{C} and \mathcal{D} have the form

$$\mathcal{C} = \left(\begin{array}{c|c} \mathcal{U} & 0 \\ \hline 0, \dots, 0 & 1 \end{array} \right) \quad \mathcal{D} = \left(\begin{array}{c|c} \mathcal{V} & 0 \\ \hline 0, \dots, 0 & 1 \end{array} \right)$$

for some integer $(n \times n)$ -matrices \mathcal{U} and \mathcal{V} . For each $i = 0, 1, \dots, n$ we have $\mathcal{C}\tilde{v}_i = \mathcal{C}(\text{den}(v_i) \cdot (v_i, 1)) = \text{den}(w_i) \cdot (\mathcal{U}v_i, 1)$. From $\mathcal{C}\tilde{v}_i = \tilde{w}_i = \text{den}(w_i) \cdot (w_i, 1)$ we obtain $\mathcal{U}v_i = w_i$. Similarly, $\mathcal{V}w_i = v_i$. In conclusion, both maps $\gamma(x) = \mathcal{U}x$ and $\mu(y) = \mathcal{V}y$ are in $\text{GL}(n, \mathbb{Z})$. Further, $\gamma^{-1} = \mu$ and $\gamma(v_i) = w_i$ for each $i \in \{0, 1, \dots, n\}$. \square

Notation. For each $w \in \mathbb{Q}^n$ we use the notation

$$\|w\|_{\mathbb{Q}} = \lambda_1(\text{conv}(0, w)),$$

where λ_1 is the 1-dimensional rational measure introduced in Subsection 3.1. Further, for every nonempty rational affine space $F \subseteq \mathbb{R}^n$ we let

$$d_F = \min\{\text{den}(v) \mid v \in F \cap \mathbb{Q}^n\}.$$

Lemma 6. For fixed $n = 1, 2, 3, \dots$ and $e = 0, \dots, n$, let F be an e -dimensional rational affine space in \mathbb{R}^n . Let $v_0 \in F \cap \mathbb{Q}^n$ with $\text{den}(v_0) = d_F$. Then there are rational points $v_1, \dots, v_e \in F$, all with denominator d_F , such that $\text{conv}(v_0, \dots, v_e)$ is a regular e -simplex.

Proof. Starting from any rational e -simplex $R \subseteq F$ having v_0 among its vertices and using [10, Lemma 2.1], one immediately obtains rational points $w_1, \dots, w_e \in F$ such that $\text{conv}(v_0, w_1, \dots, w_e)$ is regular. If $\text{den}(w_i) > d_F$ for some $i \in \{1, \dots, e\}$, then let the integer m_i be uniquely determined by writing $m_i \cdot d_F < \text{den}(w_i) \leq (m_i + 1) \cdot d_F$. Let $v_i \in F \cap \mathbb{Q}^n$ be the unique rational point with $\tilde{v}_i = \tilde{w}_i - m_i \tilde{v}_0$. Then $d_F \leq \text{den}(v_i) = \text{den}(w_i) - m_i \cdot \text{den}(v_0) = \text{den}(w_i) - m_i \cdot d_F \leq d_F$. The new $(e+1)$ -tuple of integer vectors $(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_e)$ can be completed to a basis of \mathbb{Z}^{n+1} , precisely as $(\tilde{v}_0, \tilde{w}_1, \dots, \tilde{w}_e)$ does. Thus $\text{conv}(v_0, \dots, v_e)$ is regular. \square

5. CLASSIFICATION OF RATIONAL AFFINE SUBSPACES OF \mathbb{R}^n : CONCLUSION

Generalizing the above definition of \mathcal{P}_B (Theorem 4(v)), given $v_1, \dots, v_k \in \mathbb{R}^n$ with $k \leq n$ we let $\mathcal{P}(v_1, \dots, v_k)$ denote the closed k -parallelotope generated by v_1, \dots, v_k , in symbols,

$$\mathcal{P}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^k \alpha_i v_i \in \mathbb{R}^n \mid \alpha_1, \dots, \alpha_k \in [0, 1] \right\}.$$

Lemma 7. *Let $F \subseteq \mathbb{R}^n$ be an e -dimensional rational affine space ($e = 0, \dots, n$). Let the points $v_0, \dots, v_e, w_0, \dots, w_e \in F$ satisfy $\text{den}(v_i) = \text{den}(w_i) = d_F$ for each $i = 0, \dots, e$. Suppose further that both $\text{conv}(v_0, \dots, v_e)$ and $\text{conv}(w_0, \dots, w_e)$ are regular e -simplexes. Then $\lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)) = \lambda_{e+1}(\mathcal{P}(w_0, \dots, w_e))$.*

Proof. Recall the notation of (1) for the i -dimensional part of a polyhedron. If $0 \in F$, then $(\mathcal{P}(0, v_0, \dots, v_e))^{(e+1)} = \emptyset = (\mathcal{P}(0, w_0, \dots, w_e))^{(e+1)}$, and hence

$$\lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)) = 0 = \lambda_{e+1}(\mathcal{P}(w_0, \dots, w_e)).$$

Now assume that $0 \notin F$. Let G be the linear subspace of \mathbb{R}^n generated by $F \cup \{0\}$. Then G is an $(e+1)$ -dimensional rational subspace and $d_G = 1$. By Lemma 6 there exist $z_1, \dots, z_{e+1} \in G$ such that $\text{den}(z_i) = 1$ for each $i = 1, \dots, (e+1)$ and $\text{conv}(0, z_1, \dots, z_{e+1})$ is a regular $(e+1)$ -simplex. Let $\varepsilon_1, \dots, \varepsilon_n$ be the canonical basis of the vector space \mathbb{R}^n . Lemma 5 yields $\gamma \in \text{GL}(n, \mathbb{Z})$ such that $\gamma(z_i) = \varepsilon_i$ for each $i = 1, \dots, (e+1)$. Then $\gamma(F) = F' = \mathbb{R}\varepsilon_1 + \dots + \mathbb{R}\varepsilon_{e+1}$, whence there exist $v'_0, \dots, v'_e, w'_0, \dots, w'_e \in \mathbb{R}^e$ such that $\gamma(v_i) = (v'_i, 0, \dots, 0)$ and $\gamma(w_i) = (w'_i, 0, \dots, 0)$ for each $i = 0, \dots, e$. By Theorem 4 (i) and (iii),

$$\lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)) = \lambda_{e+1}(\gamma(\mathcal{P}(v_0, \dots, v_e))) = \lambda_{e+1}(\mathcal{P}(v'_0, \dots, v'_e)). \quad (2)$$

Similarly,

$$\lambda_{e+1}(\mathcal{P}(w_0, \dots, w_e)) = \lambda_{e+1}(\mathcal{P}(w'_0, \dots, w'_e)). \quad (3)$$

From $\mathcal{P}(v'_0, \dots, v'_e), \mathcal{P}(w'_0, \dots, w'_e) \subseteq \mathbb{R}^e$ we get

$$\lambda_{e+1}(\mathcal{P}(v'_0, \dots, v'_e)) = \mathcal{H}^{e+1}(\mathcal{P}(v'_0, \dots, v'_e)) \quad (4)$$

and

$$\lambda_{e+1}(\mathcal{P}(w'_0, \dots, w'_e)) = \mathcal{H}^{e+1}(\mathcal{P}(w'_0, \dots, w'_e)) \quad (5)$$

Theorem 4(vi) yields a real $\kappa_{F'}$ such that

$$\mathcal{H}^e(\text{conv}(v'_0, \dots, v'_e)) = \kappa_{F'} \lambda_e(\text{conv}(v'_0, \dots, v'_e)) = \kappa_{F'} \frac{1}{e! d_F^{e+1}}.$$

Similarly, $\mathcal{H}^e(\text{conv}(w'_0, \dots, w'_e)) = \kappa_{F'} \frac{1}{e! d_F^{e+1}}$. Then

$$\mathcal{H}^e(\text{conv}(v'_0, \dots, v'_e)) = \mathcal{H}^e(\text{conv}(w'_0, \dots, w'_e)).$$

Since $v'_i, w'_i \in F'$ for each $i = 0, \dots, e$, we can write

$$\mathcal{H}^{e+1}(\text{conv}(0, v'_0, \dots, v'_e)) = \mathcal{H}^{e+1}(\text{conv}(0, w'_0, \dots, w'_e)). \quad (6)$$

Finally, combining (2)-(6), we obtain

$$\begin{aligned}\lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)) &= \lambda_{e+1}(\mathcal{P}(v'_0, \dots, v'_e)) = \mathcal{H}^{e+1}(\mathcal{P}(v'_0, \dots, v'_e)) \\ &= \mathcal{H}^{e+1}(\mathcal{P}(0, w'_0, \dots, w'_e)) = \lambda_{e+1}(\mathcal{P}(w'_0, \dots, w'_e)) \\ &= \lambda_{e+1}(\mathcal{P}(w_0, \dots, w_e)).\end{aligned}$$

□

Definition 8. For every $e = 0, 1, \dots, n$ and e -dimensional rational affine space $F \subseteq \mathbb{R}^n$ we let

$$\mathbf{V}_F = \lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)),$$

where v_0, \dots, v_e are rational points of F of denominator d_F such that the e -simplex $\text{conv}(v_0, \dots, v_e)$ is regular. The existence of these points is guaranteed by Lemma 6, and the independence of \mathbf{V}_F from the choice of v_0, \dots, v_e follows from Lemma 7.

As usual, given $\kappa \in \mathbb{R}$ and $S \subseteq \mathbb{R}^n$ we let $\kappa S = \{\kappa x \mid x \in S\}$.

Lemma 9. For every e -dimensional rational affine space $F \subseteq \mathbb{R}^n$ ($e = 0, \dots, n$), letting $d = d_F$ we have $\mathbf{V}_{dF} = d^{e+1}\mathbf{V}_F$.

Proof. Observe that $\mathbf{V}_F = 0$ iff $0 \in F$. If this is the case then the result follows trivially since $0 \in dF$. So assume $0 \notin F$, whence $e < n$. In view of Lemma 6, let v_0, \dots, v_e be rational points of F of denominator d such that the e -simplex $\text{conv}(v_0, \dots, v_e)$ is regular. Let $G = dF$. Then $dv_0, \dots, dv_e \in \mathbb{Z}^n \cap G$.

We claim that $\text{conv}(dv_0, \dots, dv_e)$ is regular. For otherwise, there exists $v \in \text{conv}(dv_0, \dots, dv_e) \cap \mathbb{Q}^n$ such that $\text{den}(v) \leq e$. It follows that $w = \frac{1}{d}v \in F$, $w \in \text{conv}(v_0, \dots, v_e)$ and $\text{den}(w) \leq d \text{den}(v) \leq de$, thus contradicting the regularity of $\text{conv}(v_0, \dots, v_e)$.

Let A be the $(e+1)$ -dimensional rational linear subspace of \mathbb{R}^n generated by v_0, \dots, v_e . Theorem 4 yields a constant $\kappa_A > 0$, such that $\lambda_{e+1}(Q) = \kappa_A \cdot \mathcal{H}^{e+1}(Q)$ for every rational $(e+1)$ -simplex $Q \subseteq A$. Since both $\mathcal{P}(v_0, \dots, v_e)$ and $\mathcal{P}(dv_0, \dots, dv_e)$ are finite unions of $(e+1)$ -simplexes we conclude

$$\begin{aligned}\mathbf{V}_G &= \lambda_{e+1}(\mathcal{P}(dv_0, \dots, dv_e)) = \kappa_A \cdot \mathcal{H}^{e+1}(d\mathcal{P}(v_0, \dots, v_e)) \\ &= \kappa_A d^{e+1} \cdot \mathcal{H}^{e+1}(\mathcal{P}(v_0, \dots, v_e)) = d^{e+1} \lambda_{e+1}(\mathcal{P}(v_0, \dots, v_e)) = d^{e+1} \mathbf{V}_F.\end{aligned}$$

□

Lemma 10. Every e -dimensional rational affine space $F \subseteq \mathbb{R}^n$ ($e = 0, \dots, n$), contains a point $v \in F \cap \mathbb{Q}^n$ such that $\text{den}(v) = d_F$ and $d_F^e \mathbf{V}_F = \|v\|_{\mathbb{Q}}$.

Proof. If $\mathbf{V}_F = 0$ then $0 \in F$ and $v = 0$ will do.

If $\mathbf{V}_F \neq 0$ let us first assume $d_F = 1$. By the same argument used in Lemma 7 we can assume $\dim(F) = n-1$ without loss of generality.

Let $v_0, \dots, v_{n-1} \in F \cap \mathbb{Z}^n$ be such that $\text{conv}(v_0, \dots, v_{n-1})$ is a regular e -simplex. Assume that v_0, \dots, v_{n-1} are ordered in such a way that the determinant of the matrix \mathcal{U} whose rows are v_0, \dots, v_{n-1} is > 0 . Since $\mathbf{V}_F = \lambda_n(\mathcal{P}(v_0, \dots, v_{n-1}))$, by Theorem 4(vi) we can write

$$\mathbf{V}_F = \lambda_n(\mathcal{P}(v_0, \dots, v_{n-1})) = \mathcal{H}^n(\mathcal{P}(v_0, \dots, v_{n-1})) = \det(\mathcal{U}).$$

Since $\text{conv}(v_0, \dots, v_{n-1})$ is regular, then so is $\text{conv}(0, v_1 - v_0, \dots, v_{n-1} - v_0)$. For each $i = 1, \dots, n-1$, $v_i - v_0 \in \mathbb{Z}^n$, and hence $\text{den}(v_i - v_0) = 1$. Therefore, there exists $(z_1, \dots, z_{n+1}) \in \mathbb{Z}^{n+1}$ such that

$$\{(0, \dots, 0, 1), (z_1, \dots, z_{n+1}), (v_1 - v_0, 1), \dots, (v_{n-1} - v_0, 1)\}$$

is a basis of \mathbb{Z}^{n+1} . Without loss of generality we may assume that $z_{n+1} = 1 =$ the value of the determinant of the matrix \mathcal{W} whose rows are

$$(0, \dots, 0, 1), (z_1, \dots, z_{n+1}), (v_1 - v_0, 1), \dots, (v_{n-1} - v_0, 1).$$

Let $w = (z_1, \dots, z_n)$. Since $\text{conv}(0, w)$ is regular then $\|w\|_{\mathbb{Q}} = 1$.

Claim: $\mathbf{V}_F \cdot w \in F$.

As a matter of fact:

$$\begin{aligned} \det \begin{pmatrix} (\mathbf{V}_F \cdot w) - v_0 \\ v_1 - v_0 \\ \vdots \\ v_{n-1} - v_0 \end{pmatrix} &= \mathbf{V}_F \det \begin{pmatrix} w \\ v_1 - v_0 \\ \vdots \\ v_{n-1} - v_0 \end{pmatrix} - \det \begin{pmatrix} v_0 \\ v_1 - v_0 \\ \vdots \\ v_{n-1} - v_0 \end{pmatrix} \\ &= \mathbf{V}_F \det \begin{pmatrix} 0, \dots, 0 & 1 \\ w & 1 \\ v_1 - v_0 & 1 \\ \vdots & \vdots \\ v_{n-1} - v_0 & 1 \end{pmatrix} - \det(\mathcal{U}) \\ &= \mathbf{V}_F \det(\mathcal{W}) - \det(\mathcal{U}) = 0. \end{aligned}$$

Therefore, the point $(\mathbf{V}_F \cdot w) - v_0$ lies in the $(n-1)$ -dimensional subspace of \mathbb{R}^n generated by $(v_1 - v_0), \dots, (v_{n-1} - v_0)$, that is, $\mathbf{V}_F \cdot w \in \text{aff}(v_0, \dots, v_{n-1}) = F$, which settles our claim.

Setting now $v = \mathbf{V}_F \cdot w$, we have $v \in F$ by our claim. Another application of Theorem 4(vi) yields

$$\|v\|_{\mathbb{Q}} = \lambda_1(\text{conv}(0, \mathbf{V}_F \cdot w)) = \mathbf{V}_F \lambda_1(\text{conv}(0, w)) = \mathbf{V}_F \|w\|_{\mathbb{Q}} = \mathbf{V}_F.$$

If $d_F \neq 1$, let $G = d_F F$. Observe that $d_G = 1$. Thus there exists $w \in G$ such that $\|w\|_{\mathbb{Q}} = \mathcal{V}_G$. Let $v \in F$ be such that $w = d_F v$. By Lemma 9 and Theorem 4(vi),

$$\|v\|_{\mathbb{Q}} = \|w\|_{\mathbb{Q}}/d_F = \mathcal{V}_G/d_F = d_F^e \mathbf{V}_F. \quad \square$$

We are now in a position to classify $\text{GL}(n, \mathbb{Z})$ -orbits of rational affine subspaces of \mathbb{R}^n .

Theorem 11. *Let $F, G \subseteq \mathbb{R}^n$ be nonempty rational affine spaces. Then the following conditions are equivalent:*

- (i) *There exists $\gamma \in \text{GL}(n, \mathbb{Z})$ such that $\gamma(F) = G$;*
- (ii) $(\dim(F), \mathbf{V}_F) = (\dim(G), \mathbf{V}_G)$.

Proof. The (i) \Rightarrow (ii) implication is trivial.

(ii) \Rightarrow (i) By the same argument used in Lemma 7 we can assume without loss of generality that $\dim(F) = \dim(G) = n-1$. We argue by cases:

If $\mathbf{V}_F = \mathbf{V}_G = 0$ then $0 \in F \cap G$. By Lemma 6, there exist $v_1, \dots, v_{n-1} \in F$ and $w_1, \dots, w_{n-1} \in G$ such that $\text{den}(v_i) = \text{den}(w_i) = 1$ for each $i = 1, \dots, n-1$ and both $(n-1)$ -simplexes $\text{conv}(0, v_1, \dots, v_{n-1})$ and $\text{conv}(0, w_1, \dots, w_{n-1})$ are regular. Therefore, there exist $v, w \in \mathbb{Z}^n$ such that $\text{conv}(0, v, v_1, \dots, v_{n-1})$ and $\text{conv}(0, w, w_1, \dots, w_{n-1})$ are regular. An application of Lemma 5 yields $\gamma \in \text{GL}(n, \mathbb{Z})$ such that $\gamma(v_i) = w_i$ for each $i = 1, \dots, n-1$. Thus

$$\gamma(F) = \gamma(\text{aff}(0, v_1, \dots, v_{n-1})) = \text{aff}(\gamma(0), \gamma(v_1), \dots, \gamma(v_{n-1})) = G.$$

Now assume that $\mathbf{V}_F = \mathbf{V}_G \neq 0$. Let $e = \dim(F) = \dim(G)$. By Lemma 10, there exist $v \in F$ and $w \in G$ such that

$$\|v\|_{\mathbb{Q}} = d_F^{n-1} \mathbf{V}_F \quad \text{and} \quad \|w\|_{\mathbb{Q}} = d_G^{n-1} \mathbf{V}_G. \quad (7)$$

Claim 1: $d_F = d_G$.

By (7), $d_F^{e+1}\mathbf{V}_F = \|d_F v\| = \|\text{den}(v)v\|_{\mathbb{Q}} \in \mathbb{Z}$. Moreover, if $k \in \{1, 2, \dots\}$ is such that $k^{e+1}\mathbf{V}_F \in \mathbb{Z}$, then $k(\frac{k}{d_F})^e \|v\|_{\mathbb{Q}} \in \mathbb{Z}$. Thus $k(\frac{k}{d_F})^e v \in \mathbb{Z}^n$, i.e., $k(\frac{k}{d_F})^e \in \mathbb{Z}$ and d_F is a divisor $k(\frac{k}{d_F})^e$. Therefore, d_F is a divisor of k . We have proved that

$$d_F = \min\{k \in \{1, 2, \dots\} \mid k^{e+1}\mathbf{V}_F \in \mathbb{Z}\}.$$

Similarly $d_G = \min\{k \in \{1, 2, \dots\} \mid k^{e+1}\mathbf{V}_G \in \mathbb{Z}\}$. Since by hypothesis $\mathbf{V}_F = \mathbf{V}_G$, then $d_F = d_G$, with settles our claim.

By (7) and Claim 1, $\|v\|_{\mathbb{Q}} = d_F^{n-1}\mathbf{V}_F = d_G^{n-1}\mathbf{V}_G = \|w\|_{\mathbb{Q}}$. By Lemma 6, there exist $v_1, \dots, v_{n-1} \in F$ and $w_1, \dots, w_{n-1} \in G$ such that $\text{den}(v_i) = \text{den}(w_i) = d_F$ for each $i = 1, \dots, n-1$ and both $(n-1)$ -simplexes $\text{conv}(v, v_1, \dots, v_{n-1})$ and $\text{conv}(w, w_1, \dots, w_{n-1})$ are regular.

Let us set $v' = v/(\|v\|_{\mathbb{Q}})$ and $w' = w/(\|w\|_{\mathbb{Q}})$. By Theorem 4(vi), $\|v'\|_{\mathbb{Q}} = \|w'\|_{\mathbb{Q}} = 1$, whence $v', w' \in \mathbb{Z}^n$.

Claim 2: $\text{conv}(0, v', v_1, \dots, v_{n-1})$ and $\text{conv}(0, w', w_1, \dots, w_{n-1})$ are regular.

As a matter of fact, again by Theorem 4(vi) we can write

$$\begin{aligned} \det \begin{pmatrix} 0 & \dots & 0 & 1 \\ & v' & & 1 \\ & d_F v_1 & & d_F \\ & \vdots & & \vdots \\ & d_F v_{n-1} & & d_F \end{pmatrix} &= \det \begin{pmatrix} v' \\ d_F v_1 \\ \vdots \\ d_F v_{n-1} \end{pmatrix} = d_F^{n-1} \det \begin{pmatrix} v' \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \\ &= \frac{d_F^{n-1}}{\|v\|_{\mathbb{Q}}} \det \begin{pmatrix} v \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \frac{d_F^{n-1}\mathbf{V}_F}{\|v\|_{\mathbb{Q}}} = 1 \end{aligned}$$

The same argument proves that $\text{conv}(0, w', w_1, \dots, w_{n-1})$ is regular, which settles our claim.

Another application of Lemma 5 yields $\gamma \in \text{GL}(n, \mathbb{Z})$ such that $\gamma(v') = w'$ and $\gamma(v_i) = w_i$ for each $i = 1, \dots, n-1$. Then

$$\gamma(v) = \gamma(\|v\|_{\mathbb{Q}} v') = \|v\|_{\mathbb{Q}} \gamma(v') = \|w\|_{\mathbb{Q}} w' = w,$$

and

$$\gamma(F) = \gamma(\text{aff}(v, v_1, \dots, v_{n-1})) = \text{aff}(\gamma(v), \gamma(v_1), \dots, \gamma(v_{n-1})) = G. \quad \square$$

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